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The Lasker–Noether Theorem in the category $\mathscr{U}(\mathrm{H}^*)$

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Abstract

We prove the Lasker–Noether Theorem in the category $\mathscr{U}(\mathrm{H}^*)$ of unstable $\mathrm{H}^* \odot \mathscr{P}^*$-modules. Along the way, we generalize Lam's $\mathscr{J}$-functor to the context of modules. © 2001 Elsevier Science B.V. All rights reserved.

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Let $\mathbb{F}$ be a Galois field of characteristic p with q elements. Consider a faithful representation of degree n

$$\rho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$$

of a finite group G. Then G acts via ρ on the vector space $V = \mathbb{F}^n$, and hence on the ring of polynomial functions

$$\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$$

in n variables via

$$gf(v) = f(\rho(g)^{-1}v) \quad \forall f \in \mathbb{F}[V],\ v \in V,\ g \in G.$$

The ring of polynomials invariant under this action is denoted by $\mathbb{F}[V]^G$. By a classical theorem of Emmy Noether, any ring of invariants $\mathbb{F}[V]^G$ is Noetherian (See [9]). Since the ground field $\mathbb{F}$ is finite, the full general linear group $\mathrm{GL}(n, \mathbb{F})$ is finite, and is

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moreover present in every ring of invariants $\mathbb{F}[V]^G$. In 1911 Dickson proved that the ring of invariants of $\mathrm{GL}(n, \mathbb{F})$,

$$\mathbb{F}[V]^{\mathrm{GL}(n,\mathbb{F})} = \mathbb{F}[\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}] = \mathscr{D}^*(n),$$

is a polynomial ring in $\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}$, which are called the *Dickson classes* (See [2]). The algebra $\mathscr{D}^*(n)$ is called the *Dickson algebra*.

As described in Chapters 10 and 11 in [10], or in the introduction of [6], $\mathbb{F}[V]^G$ inherits from $\mathbb{F}[V]$ an unstable action of the Steenrod algebra $\mathscr{P}^*$. In other words, $\mathbb{F}[V]^G$ is an object in the category $\mathscr{K}_{fg}$ of finitely generated unstable (graded connected commutative $\mathbb{F}$-) algebras over $\mathscr{P}^*$. By the Imbedding Theorem 8.1.5 in [6] *every* object H^* in $\mathscr{K}_{fg}$ contains a fractal [1] of the Dickson algebra such that

$$\mathscr{D}^*(n)^{q^t} \hookrightarrow \mathrm{H}^*$$

is an integral extension, where t can be choosen to be zero, if H^* is $\mathscr{P}^*$-inseparably closed. Therefore every finitely generated unstable $\mathbb{F}$-algebra over the Steenrod algebra can be considered as a module over $\mathscr{D}^*(n)^{q^t}$, i.e., as a finitely generated module over a Noetherian ring. In classical theory every such module has a primary decomposition. In this paper we prove the $\mathscr{P}^*$-invariant version of this statement in its most general form: Let H^* be an unstable Noetherian algebra over the Steenrod algebra. Let M be a Noetherian unstable H^*-modules. Then M has a primary decomposition

$$M = Q_1 \cap \cdots \cap Q_m,$$

consisting of unstable primary components $Q_1, \ldots, Q_m$. Moreover, the associated prime ideals

$$\mathscr{R}ad(Q_i : M) \subseteq \mathrm{H}^* \quad \forall i = 1, \ldots, m,$$

are $\mathscr{P}^*$-invariant ideals, i.e., ideals that are closed under the action of the Steenrod algebra.

This solves a long open problem (see [8] and Section 6 in [11]) that has some surprising immediate applications (see [7]).

1. Lam's $\mathscr{J}$ for modules

Let H^* be an unstable algebra over the Steenrod algebra. Note that, in this first section, we do not need to assume that the ground ring H^* is Noetherian. We denote

[1] Recall that a *fractal* of the Dickson algebra is

$$\mathscr{D}^*(n)^{q^t} = \mathbb{F}[\mathbf{d}_{n,0}^{q^t}, \ldots, \mathbf{d}_{n,n-1}^{q^t}] = \mathbb{F}[x_1^{q^t}, \ldots, x_n^{q^t}]^{\mathrm{GL}(n,\mathbb{F})}.$$

by $\mathscr{U}(\mathrm{H}^*)$ the category of unstable $\mathrm{H}^* \odot \mathscr{P}^*$ -modules. The semitensor product [2] was introduced by Massey and Peterson [4, Definition 2.5]. It summarizes that we are looking at objects M that are

(1) left H^*-modules, as well as,

(2) unstable left modules over the Steenrod algebra $\mathscr{P}^*$, and

both structures are compatible in the sense that

$$\mathscr{P}^l(hm) = \sum_{i+k=l} \mathscr{P}^i(h)\mathscr{P}^k(m)$$

for every element $\mathscr{P}^l \in \mathscr{P}^*$, $h \in \mathrm{H}^*$ and $m \in M$. In other words, the map

$$\mathrm{H}^* \otimes M \to M,$$

defining the H^*-module structure on M, is a homomorphism of left $\mathscr{P}^*$-modules.

Recall that the category $\mathscr{U}(\mathrm{H}^*)$ is abelian. In this section we want to generalize Lam's $\mathscr{I}$-functor to the context of modules (see [3] or Section 11.2 in [10]). Since this functor played a significant role in the proof of the Lasker–Noether Theorem for ideals (See [8]), it should not surprise that we need it here also.

Let M be an object in $\mathscr{U}(\mathrm{H}^*)$. Denote by $\mathscr{M}od_{\mathrm{H}^*}$ the category of H^*-modules, and by $\mathscr{M}od_{\mathrm{H}^*}(M)$ its full subcategory of H^*-submodules of M. Let N be an object in $\mathscr{M}od_{\mathrm{H}^*}(M)$. By restriction, we can define the images of the Steenrod powers on the elements of N. However, the module N might not be closed under this action. This motivates the following definition.

Definition and Lemma 1.1. *Let M be an unstable $\mathrm{H}^* \odot \mathscr{P}^*$-module. Let N be an object in $\mathscr{M}od_{\mathrm{H}^*}(M)$. We define*

$$\mathscr{I}(N) := \{n \in N \mid \mathscr{P}^i(n) \in N \ \ \forall i \geqslant 0\},$$

and iteratively

$$\mathscr{I}_j(N) = \mathscr{I}(\mathscr{I}_{j-1}(N)) \quad \text{for } j \geqslant 2.$$

This leads to a descending chain

$$\mathscr{I}_0(N) := N \supseteq \mathscr{I}_1(N) \supseteq \mathscr{I}_2(N) \supseteq \cdots$$

of H^*-modules in $\mathscr{M}od_{\mathrm{H}^*}(M)$. We denote the intersection of this chain by

$$\mathscr{I}_\infty(N) = \bigcap_{j \geqslant 0} \mathscr{I}_j(N).$$

Then $\mathscr{I}_\infty(N)$ is an unstable $\mathrm{H}^* \odot \mathscr{P}^*$-module, and moreover, the maximal H^*-submodule in N that is closed under the action of the Steenrod algebra.

[2] The multiplication in the semitensor product is defined as follows:

$$(h \otimes \mathscr{P}^l)(h' \otimes \mathscr{P}^k) := \sum_{i+j=l} h\mathscr{P}^i(h') \otimes \mathscr{P}^j\mathscr{P}^k$$

(see (2.3) or (2.4) in [4]).

Proof. For $n_1, n_2 \in \mathscr{J}_j(N)$ and $h_1, h_2 \in \mathrm{H}^*$, we have

$$\begin{aligned}\mathscr{P}^l(h_1n_1 + h_2n_2) &= \mathscr{P}^l(h_1n_1) + \mathscr{P}^l(h_2n_2)\\ &= \sum_{i+k=l}(\mathscr{P}^i(h_1)\mathscr{P}^k(n_1) + \mathscr{P}^i(h_2)\mathscr{P}^k(n_2)),\end{aligned}$$

where we made use of the Cartan formulae. Since

$$\mathscr{P}^i(h_1),\ \mathscr{P}^i(h_2) \in \mathrm{H}^* \quad \forall i \geqslant 0$$

and

$$\mathscr{P}^k(n_1), \mathscr{P}^k(n_2) \in \mathscr{J}_{j-1}(N) \quad \forall k \geqslant 0,$$

by definition of $\mathscr{J}_j(N)$, we see that

$$\mathscr{P}^l(h_1n_1 + h_2n_2) \in \mathscr{J}_{j-1}(N) \quad \forall l \geqslant 0.$$

This in turn means that

$$h_1n_1 + h_2n_2 \in \mathscr{J}_j(N),$$

making $\mathscr{J}_j(N)$ into an H^*-module. Therefore we have a chain of H^*-modules in $\mathscr{M}od_{\mathrm{H}^*}(M)$:

$$N =: \mathscr{J}_0(N) \supseteq \mathscr{J}_1(N) \supseteq \mathscr{J}_2(N) \supseteq \cdots \supseteq \mathscr{J}_j(N) \supseteq \cdots \supseteq \mathscr{J}_\infty(N).$$

Finally, we need to show that $\mathscr{J}_\infty(N)$ is closed under the action of the Steenrod algebra. To this end, let $n \in \mathscr{J}_\infty(N)$, then

$$n \in \mathscr{J}_j(N) \quad \forall j \geqslant 0.$$

Hence

$$\mathscr{P}^l(n) \in \mathscr{J}_{j-1}(N) \quad \forall j \geqslant 1,\ \forall l \geqslant 0,$$

i.e.,

$$\mathscr{P}^l(n) \in \mathscr{J}_\infty(N) \quad \forall l \geqslant 0,$$

as claimed. The maximality of $\mathscr{J}_\infty(N)$ is by construction clear. $\square$

Let $I = (i_1, \ldots, i_k)$ be a multi index, and set $\mathscr{P}^I = \mathscr{P}^{i_1} \cdots \mathscr{P}^{i_k}$. Then an element $n \in N$ is in $\mathscr{J}_\infty(N)$ if and only if

$$\mathscr{P}^I(n) \in N \quad \forall \text{multi index } I.$$

In the following series of technical lemmata we show that the category $\mathscr{U}(\mathrm{H}^*)$ of unstable $\mathrm{H}^* \odot \mathscr{P}^*$-modules is closed under certain standard module-theoretic operations. Moreover, we investigate the behavior of such operations under the $\mathscr{J}_\infty$-functor. Needless to say, we do this, because we will use these results later on.

Lemma 1.2. *Let N, N' be objects in $\mathscr{M}od_{\mathrm{H}^*}(M)$, and M in $\mathscr{U}(\mathrm{H}^*)$. Then*

$$\mathscr{J}_\infty(N \cap N') = \mathscr{J}_\infty(N) \cap \mathscr{J}_\infty(N').$$

Proof. Take an element $n \in \mathscr{J}_\infty(N \cap N')$ then

$$n \in N \cap N' \text{ and } \mathscr{P}^I(n) \in N \cap N' \quad \text{for all multi indeces } I.$$

This means

$$n \in \mathscr{J}_\infty(N) \cap \mathscr{J}_\infty(N'),$$

establishing the inclusion "$\subseteq$". The converse inclusion is proved by using this argument backward. □

Lemma 1.3. *Let M and M' be objects in $\mathscr{U}(\mathrm{H}^*)$. Then so is their quotient $(M' : M)$.*

Proof. The quotient $(M' : M) \subseteq \mathrm{H}^*$ is an ideal in H^*, and we need to show that it is $\mathscr{P}^*$-invariant. So, take an element $h \in (M' : M)$, i.e., $h \cdot M \subseteq M'$. We claim that $\mathscr{P}^i(h) \in (M' : M)$, i.e., we claim that

$$\mathscr{P}^i(h) \cdot M \subseteq M'$$

for every $i \geqslant 0$. We proceed by induction on i. Since $\mathscr{P}^0$ is the identity map, the case $i = 0$ is trivial. However, we need to start our induction with $i = 1$. Let $m \in M$. Then we have by the Cartan formulae

$$\mathscr{P}^1(h)m = \mathscr{P}^1(hm) - h\mathscr{P}^1(m).$$

Now, the first summand $\mathscr{P}^1(hm) \in M'$, because $hm \in M'$ and M' is closed under the action of the Steenrod algebra. The second summand, $h\mathscr{P}^1(m)$, is equally in M', because $m \in M$, therefore $\mathscr{P}^1(m) \in M$, and $h \in (M' : M)$. Hence

$$\mathscr{P}^1(h)m \in M'$$

for every $m \in M$. This means that

$$\mathscr{P}^1(h) \in (M' : M).$$

Let $i > 1$. Then the Cartan formulae tell us that

$$\mathscr{P}^i(h)m = \mathscr{P}^i(hm) - \sum_{k+l=i, k<i} \mathscr{P}^k(h)\mathscr{P}^l(m) \quad \forall m \in M.$$

The sum on the right-hand side is by induction in M'. Since M' is in $\mathscr{U}(\mathrm{H}^*)$, and therefore $\mathscr{P}^i(hm) \in M'$, we conclude that also $\mathscr{P}^i(h)m \in M'$, in other words

$$\mathscr{P}^i(h) \in (M' : M) \quad \forall i \geqslant 0,$$

as claimed. □

Lemma 1.4. *Let M be an object in $\mathscr{U}(\mathrm{H}^*)$, N in $\mathscr{M}od_{\mathrm{H}^*}(M)$. Then*

$$(\mathscr{J}_\infty(N) : M) = \mathscr{J}_\infty(N : M).$$

Proof. Since $\mathscr{I}_\infty(N) \subseteq N$, we have

$$\begin{aligned}(\mathscr{I}_\infty(N):M) &:= \{h \in \mathrm{H}^* \mid hM \subseteq \mathscr{I}_\infty(N)\}\\ &\subseteq \{h \in \mathrm{H}^* \mid hM \subseteq N\} =: (N:M).\end{aligned}$$

Since $\mathscr{I}_\infty(N)$ and M are $\mathscr{P}^*$-modules, so is their quotient, by the preceding Lemma 1.3. Therefore

$$(\mathscr{I}_\infty(N):M) = \mathscr{I}_\infty(\mathscr{I}_\infty(N):M) \subseteq \mathscr{I}_\infty(N:M),$$

by maximality of $\mathscr{I}_\infty(N:M)$ in $(N:M)$. This establishes the inclusion "$\subseteq$". To show the reverse inclusion, we take an element $h \in \mathscr{I}_\infty(N:M)$. Then

$$hM \subseteq N \text{ and } \mathscr{P}^I(h)M \subseteq N \quad \text{for all multi indeces } I.$$

We need to show that $h \in (\mathscr{I}_\infty(N):M)$, i.e., $hM \subseteq \mathscr{I}_\infty(N)$. Since $hM \subseteq N$, this means for every multi index I and every element $m \in M$ we have to verify that

$$\mathscr{P}^I(hm) \in N.$$

We employ the Cartan formulae, set $I = (i_1, \ldots, i_k)$ and get

$$\mathscr{P}^I(hm) = \mathscr{P}^{I'}\left(\sum_{j_k+l_k=i_k} \mathscr{P}^{j_k}(h)\mathscr{P}^{l_k}(m)\right),$$

where $I' = (i_1, \ldots, i_{k-1})$. Hence, setting $I'' = (i_1, \ldots, i_{k-2})$ and iterating, we arrive at

$$\begin{aligned}\mathscr{P}^I(hm) &= \mathscr{P}^{I'}\left(\sum_{j_k+l_k=i_k} \mathscr{P}^{j_k}(h)\mathscr{P}^{l_k}(m)\right)\\ &= \mathscr{P}^{I''}\left(\sum_{j_{k-1}+l_{k-1}=i_{k-1}} \sum_{j_k+l_k=i_k} \mathscr{P}^{j_{k-1}}\mathscr{P}^{j_k}(h)\mathscr{P}^{l_{k-1}}\mathscr{P}^{l_k}(m)\right)\\ &= \sum_{j_1+l_1=i_1} \cdots \sum_{j_k+l_k=i_k} \mathscr{P}^{j_1}\cdots\mathscr{P}^{j_k}(h)\mathscr{P}^{l_1}\cdots\mathscr{P}^{l_k}(m)\\ &= \sum_{j_1+l_1=i_1} \cdots \sum_{j_k+l_k=i_k} \mathscr{P}^J(h)\mathscr{P}^L(m),\end{aligned}$$

for multi indices $J = (j_1, \ldots, j_k)$ and $L = (l_1, \ldots, l_k)$. Since M is an object in $\mathscr{U}(\mathrm{H}^*)$, we have $\mathscr{P}^L(m) \in M$ for all L. Now, $h \in \mathscr{I}_\infty(N:M)$, i.e.,

$$hm \in N \quad \forall m \in M \quad \text{and} \quad \mathscr{P}^J(h)m \in N \quad \forall m \in M, \ \forall J.$$

This means that the right-hand side is an element of N, hence so is the left as desired. □

We collect these results, and extend them to

Proposition 1.5. *Let M, M' be unstable $\mathrm{H}^* \odot \mathscr{P}^*$-modules, let N be an object in $\mathscr{M}od_{\mathrm{H}^*}(M)$. Then*
(1) $(M' : M) \subseteq \mathrm{H}^*$ *is $\mathscr{P}^*$-invariant.*
(2) *If $(N : M) \subseteq \mathrm{H}^*$ is $\mathscr{P}^*$-invariant, then there exists an unstable $\mathrm{H}^* \odot \mathscr{P}^*$-module $M_N \subseteq M$ such that $(M_N : M) = (N : M)$.*

Proof. Statement (1) is the contents of Lemma 1.3. To prove the second statement recall from Lemma 1.4 that

$$(N : M) = \mathscr{J}_\infty(N : M) = (\mathscr{J}_\infty(N) : M).$$

Choose $M_N = \mathscr{J}_\infty(N)$ and finish the proof with Lemma 1.1. □

Denote by $\mathscr{R}ad(-)$ the radical of $(-)$. We need the following result correlating the functor $\mathscr{J}_\infty$ with $\mathscr{R}ad$.

Lemma 1.6. *Let M be an object in $\mathscr{U}(\mathrm{H}^*)$, and let N an object in $\mathscr{M}od_{\mathrm{H}^*}(M)$. Then* [3]

$$\mathscr{R}ad(\mathscr{J}_\infty(N) : M) = \mathscr{J}_\infty(\mathscr{R}ad(N : M)).$$

Proof. Let $h \in \mathscr{R}ad(\mathscr{J}_\infty(N) : M)$. Then

$$h^r M \subseteq \mathscr{J}_\infty(N)$$

for some large $r \in \mathbb{N}$. Hence, a fortiori,

$$h^r M \subseteq N, \quad \text{or} \quad h \in \mathscr{R}ad(N : M).$$

This means

$$\mathscr{R}ad(\mathscr{J}_\infty(N) : M) \subseteq \mathscr{R}ad(N : M).$$

By Lemma 1.3 $(\mathscr{J}_\infty(N) : M)$ is $\mathscr{P}^*$-invariant. Therefore so is its radical by Lemma 1.4 in [8]. By maximality this implies that

$$\mathscr{R}ad(\mathscr{J}_\infty(N) : M) \subseteq \mathscr{J}_\infty(\mathscr{R}ad(N : M)).$$

To show the reverse inclusion recall that we have

$$\mathscr{J}_\infty(\mathscr{R}ad(N : M)) = \mathscr{R}ad(\mathscr{J}_\infty(N : M))$$

by Lemma 1.3 in [8]. We want to show that $\mathscr{R}ad(\mathscr{J}_\infty(N : M)) \subseteq \mathscr{R}ad(\mathscr{J}_\infty(N) : M)$. So it is enough to show that

$$\mathscr{J}_\infty(N : M) \subseteq (\mathscr{J}_\infty(N) : M).$$

[3] To be precise we should have to write $\mathscr{R}ad((N : M))$, because we are taking the radical of the ideal $(N : M)$. However, we omit the second set of parentheses and write $\mathscr{R}ad(N : M)$, because this is better to read.

For that take an element $h \in \mathscr{I}_\infty(N : M)$. Then by definition we have that

$$h \cdot M \subseteq N \text{ and } \mathscr{P}^I(h) \cdot M \subseteq N$$

for every multi index $I = (i_1, \ldots, i_k)$. Let $m \in M$. We need to show that $h \cdot m \in \mathscr{I}_\infty(N)$. In otherwords, we need to show that

$$\mathscr{P}^I(h \cdot m) \in N \quad \text{for all multi indeces } I.$$

We induct on the length $k = |I|$, where $I = (i_1, \ldots, i_k)$.

Case $|I| = 1$: Then $I = (i_1) = (i)$ and

$$\mathscr{P}^i(hm) = \sum_{k+l=i} \mathscr{P}^k(h)\mathscr{P}^l(m).$$

Since $h \in \mathscr{I}_\infty(N : M)$ we have that $\mathscr{P}^k(h) \in \mathscr{I}_\infty(N : M) \subseteq (N : M)$ for all k. Also, because $\mathscr{P}^l(m) \in M$ for all l, and hence

$$\mathscr{P}^k(h) \cdot \mathscr{P}^l(m) \in N \quad \forall k, \ \forall l.$$

Therefore

$$\mathscr{P}^i(hm) = \sum_{k+l=i} \mathscr{P}^k(h)\mathscr{P}^l(m) \in N.$$

Case $|I| > 1$: We rewrite $I = (I', i)$, where $I' = (i_1, \ldots, i_{k-1})$, $I = (i_1, \ldots, i_k)$ and $i = i_k$. We have

$$\begin{aligned}\mathscr{P}^I(h \cdot m) &= \mathscr{P}^{I'}\mathscr{P}^i(h \cdot m)\\ &= \mathscr{P}^{I'}\left(\sum_{k+l=i} \mathscr{P}^k(h)\mathscr{P}^l(m)\right)\\ &= \sum_{k+l=i} \mathscr{P}^{I'}(\mathscr{P}^k(h)\mathscr{P}^l(m))\end{aligned}$$

As in the preceding case we conclude that

$$\mathscr{P}^l(m) \in M \quad \forall l \geqslant 0, \quad \text{and} \quad \mathscr{P}^k(h) \in \mathscr{I}_\infty(N : M) \quad \forall k \geqslant 0.$$

Hence by induction we have

$$\mathscr{P}^{I'}(\mathscr{P}^k(h)\mathscr{P}^l(m)) \in N$$

and we are done. $\square$

2. Primary unstable $\mathrm{H}^* \odot \mathscr{P}^*$-modules

We want to show that an unstable Noetherian module over an unstable Noetherian algebra, H^*, has a primary decomposition consisting of unstable components. For that we follow the classical route as described in the appendix to Chapter IV on p. 252f of [12]. We start with recollecting some terminology. Let H^* be an unstable Noetherian algebra over the Steenrod algebra. Let M be an unstable $\mathrm{H}^* \odot \mathscr{P}^*$-module, and $Q \subseteq M$ a submodule. Q is said to be *primary*, if whenever

$$h \cdot m \in Q \quad \text{for } h \in \mathrm{H}^*,\ m \in M$$

then

$$\text{either} \quad m \in Q \quad \text{or} \quad h \in \mathscr{R}ad(Q) := \mathscr{R}ad(Q : M).$$

If a module $Q \subseteq M$ is primary, then the ideal

$$(Q : M) := \{h \in \mathrm{H}^* \mid hM \subseteq Q\} \subseteq \mathrm{H}^*$$

is primary (but not conversely!).[4] Let M be Noetherian, and let $M' \subseteq M$ be unstable $\mathrm{H}^* \odot \mathscr{P}^*$-modules. As an H^*-submodule of M, the module M' has a primary decomposition

$$M' = Q_1 \cap \cdots \cap Q_m,$$

where $Q_1, \ldots, Q_m \subseteq M$ are primary H^*-submodules of M. The prime ideals

$$\mathfrak{p}_i = \mathscr{R}ad(Q_i : M) \subset \mathrm{H}^* \quad \forall i = 1, \ldots, m$$

are called *associated prime ideals* of M'. Moreover, the decomposition is called *irredundant* if

$$\bigcap_{i \neq j} Q_i \not\subset Q_j \quad \forall j = 1, \ldots, m.$$

It is called *minimal* if

$$\mathscr{R}ad(Q_i : M) \neq \mathscr{R}ad(Q_j : M) \quad \text{whenever } i \neq j.$$

We want to show that the primary modules $Q_1, \ldots, Q_m$ as well as the associated prime ideals can be choosen to be $\mathscr{P}^*$-invariant. We start by showing that the associated prime ideals are $\mathscr{P}^*$-invariant.

Lemma 2.1. *Let M be Noetherian. Let $M' \subseteq M$ be objects in $\mathscr{U}(\mathrm{H}^*)$, let*

$$M' = Q_1 \cap \cdots \cap Q_m$$

be a primary decomposition of M' as an H^*-module. Then the associated prime ideals of M',

$$\mathfrak{p}_i := \mathscr{R}ad(Q_i : M) \subseteq \mathrm{H}^* \quad \forall i = 1, \ldots, m,$$

are $\mathscr{P}^*$-invariant.

Proof. Since $M' \subseteq M$ are objects in $\mathscr{U}(\mathrm{H}^*)$, the ideal $(M' : M)$ in H^* is $\mathscr{P}^*$-invariant by Lemma 1.3. By Theorem 3.5 in [8] the ideal $(M' : M)$ has a $\mathscr{P}^*$-invariant minimal irredundant primary decomposition

$$(M' : M) = \widehat{\mathfrak{q}_1} \cap \cdots \cap \widehat{\mathfrak{q}_k}$$

[4] See p. 252 in [12] for a counterexample.

with associated prime components

$$\widehat{\mathfrak{p}_j} = \mathscr{R}ad(\widehat{\mathfrak{q}_j}) \subseteq \mathrm{H}^*,$$

for $j = 1, \ldots, k$. Hence

$$\begin{aligned}\widehat{\mathfrak{p}_1} \cap \cdots \cap \widehat{\mathfrak{p}_k} &= \mathscr{R}ad(M' : M)\\ &= \mathscr{R}ad(Q_1 : M) \cap \cdots \cap \mathscr{R}ad(Q_m : M)\\ &= \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m.\end{aligned}$$

Since both decompositions are minimal and irredundant we have

$$k = m,$$

and, after possibly reordering,

$$\widehat{\mathfrak{p}_i} = \mathscr{R}ad(Q_i : M) \quad \forall i = 1, \ldots, k$$

(cf. Lemma 2.4.10 in [1]). $\square$

The following result extends Lemma 1.5 in [8] to the context of modules.

Proposition 2.2. *Let M be an unstable Noetherian $\mathrm{H}^* \odot \mathscr{P}^*$-module. Let Q in $\mathscr{M}od_{\mathrm{H}^*}(M)$ be a primary module, such that its radical*

$$\mathfrak{p} = \mathscr{R}ad(Q : M) \subseteq \mathrm{H}^*$$

is $\mathscr{P}^*$-invariant. Then $\mathscr{I}_\infty(Q)$ is primary in $\mathscr{U}(\mathrm{H}^*)$ with radical $\mathfrak{p}$.

Proof. Let Q be primary as an H^*-submodule of M. Then

$$\mathfrak{q} = (Q : M) \subseteq \mathrm{H}^*$$

is a primary ideal with radical $\mathfrak{p} \subseteq \mathrm{H}^*$. By Theorem 3.3 in [5] we know that $\mathscr{I}_\infty(\mathfrak{q})$ is a primary ideal with radical $\mathscr{I}_\infty(\mathfrak{p}) = \mathfrak{p}$. Hence, by Proposition 1.5

$$(\mathscr{I}_\infty(Q) : M) = \mathscr{I}_\infty(Q : M)$$

is a primary ideal with radical $\mathfrak{p}$. We need to show that $\mathscr{I}_\infty(Q) \subseteq M$ is a primary module. We have

$$\mathfrak{p} = \mathscr{I}_\infty(\mathfrak{p}) = \mathscr{I}_\infty(\mathscr{R}ad(Q : M)) = \mathscr{R}ad(\mathscr{I}_\infty(Q) : M) \supseteq (\mathscr{I}_\infty(Q) : M).$$

Let $h \in \mathrm{H}^*$, $m \in M$ and $hm \in \mathscr{I}_\infty(Q)$. We assume that $m \notin \mathscr{I}_\infty(Q)$, and have to show that

$$h \in \mathscr{R}ad(\mathscr{I}_\infty(Q) : M) = \mathfrak{p}.$$

Note that we have a chain of modules

$$\mathscr{I}_\infty(Q) \subseteq \mathscr{I}_1(Q) \subseteq Q$$

with the same ($\mathscr{P}^*$-invariant) prime radical

$$\mathfrak{p} = \mathscr{R}ad(Q) = \mathscr{R}ad(\mathscr{I}_1(Q)) = \mathscr{I}_\infty(\mathscr{R}ad(Q)).$$

Therefore, by iteration it is enough to show that $\mathscr{I}_1(Q)$ is primary. To this end let $h \in \mathrm{H}^*$ and $m \in M$ such that $hm \in \mathscr{I}_1(Q)$, and assume that $m \notin \mathscr{I}_1(Q)$. We need to show that

$$h \in \mathscr{R}ad(\mathscr{I}_1(Q) : M) = \mathfrak{p}.$$

Case $m \in Q \setminus \mathscr{I}_1(Q)$: Then there exists an $i \in \mathbb{N}_0$ such that $\mathscr{P}^i(m) \notin Q$. Let i be minimal with this property. Then

$$\mathscr{P}^i(hm) = \sum_{k+l=i, l<i} \mathscr{P}^k(h)\mathscr{P}^l(m) + h\mathscr{P}^i(m).$$

Since $hm \in \mathscr{I}_1(Q)$ we have that $\mathscr{P}^i(hm) \in Q$. By minimality of i we know that $\mathscr{P}^l(m) \in Q$ for all $l < i$. Therefore

$$h\mathscr{P}^i(m) = \mathscr{P}^i(hm) - \sum_{k+l=i, l<i} \mathscr{P}^k(h)\mathscr{P}^l(m) \in Q.$$

Since $\mathscr{P}^i(m) \notin Q$ by assumption and Q is primary we conclude that

$$\begin{aligned} h &\in \mathscr{R}ad(Q : M) \\ &= \mathscr{I}_\infty(\mathscr{R}ad(Q : M)) \\ &= \mathscr{R}ad(\mathscr{I}_\infty(Q) : M) \\ &= \mathfrak{p} \\ &= \mathscr{R}ad(\mathscr{I}_1(Q) : M), \end{aligned}$$

where we used Lemmas 2.1 and 1.6.

Case $m \notin Q$: We assume that $hm \in \mathscr{I}_1(Q) \subseteq Q$. Because Q is primary it follows that

$$h \in \mathscr{R}ad(Q : M) = \mathfrak{p}.$$

This shows that

$$\mathscr{I}_1(Q) \subseteq M$$

is a primary H^*-module. Hence, iteratively, we get that $\mathscr{I}_j(Q) \subseteq M \forall j$ is primary in the category $\mathscr{U}(\mathrm{H}^*)$. Finally, if $hm \in \mathscr{I}_\infty(Q)$ and $m \notin \mathscr{I}_\infty(Q)$, then $hm \in \mathscr{I}_j(Q) \forall j$ and there exists a $\mathscr{I}_0 \in \mathbb{N}_0$ such that $m \in \mathscr{I}_{j_0}(Q)$. Then $h \in \mathscr{R}ad(\mathscr{I}_{j_0}(Q) : M) = \mathscr{R}ad(\mathscr{I}_\infty(Q) : M) = \mathfrak{p}$ as desired. □

Theorem 2.3 (Lasker–Noether Theorem). *Let* H^* *be an unstable Noetherian algebra over the Steenrod algebra. Let* M *be Noetherian, and let* $M' \subseteq M$ *be unstable* $\mathrm{H}^* \odot \mathscr{P}^*$*-modules. Then* M' *admits a minimal irredundant primary decomposition in* $\mathscr{U}(\mathrm{H}^*)$, *i.e., all primary components, as well as the associated prime ideals are unstable* $\mathrm{H}^* \odot \mathscr{P}^*$*-modules.*

Proof. Choose a primary decomposition of M' as a H^*-module

$$M' = Q_1 \cap \cdots \cap Q_m.$$

By Lemma 2.1 we know that the associated prime ideals are $\mathscr{P}^*$-invariant. Hence by Proposition 2.2 we have that $\mathscr{I}_\infty(Q_i)$ is a primary module with radical

$$\mathscr{Rad}(Q_i) := \mathscr{Rad}(Q_i : M) = \mathscr{I}_\infty(\mathscr{Rad}(Q_i)).$$

By Lemma 1.1 these are modules in $\mathscr{U}(\mathrm{H}^*)$. So with the help of Lemma 1.2 we find that

$$M' = \mathscr{I}_\infty(M') = \mathscr{I}_\infty(Q_1 \cap \cdots \cap Q_m) = \mathscr{I}_\infty(Q_1) \cap \cdots \cap \mathscr{I}_\infty(Q_m)$$

is a primary decomposition in the category $\mathscr{U}(\mathrm{H}^*)$. We make it irredundant by throwing away superfluous modules and minimal by combining these modules which have the same radical. □

Corollary 2.4. *Let* H^* *be an unstable Noetherian algebra over the Steenrod algebra. Let* M *be an unstable Noetherian* $\mathrm{H}^* \odot \mathscr{P}^*$*-module. Let* Q *in* $\mathscr{Mod}_{\mathrm{H}^*}(M)$ *be a primary module, then* $\mathscr{I}_\infty(Q)$ *is primary in* $\mathscr{U}(\mathrm{H}^*)$ *with radical* $\mathscr{I}_\infty(\mathscr{Rad}(Q : M))$.

Proof. Let Q be a primary module in $\mathscr{Mod}_{\mathrm{H}^*}(M)$. Then, by definition, the ideal

$$(Q : M) \subseteq \mathrm{H}^*$$

is primary. From Theorem 3.3 in [5], we know that

$$\mathscr{I}_\infty(Q : M) \subseteq \mathrm{H}^*$$

is $\mathscr{P}^*$-invariant and primary with $\mathscr{P}^*$-invariant radical

$$\mathscr{I}_\infty(\mathscr{Rad}(Q : M)) = \mathscr{Rad}(\mathscr{I}_\infty(Q) : M),$$

where the last equality follows from Lemma 1.6. We need to show that $\mathscr{I}_\infty(Q)$ is a primary module. To this end let

$$\mathscr{I}_\infty(Q) = Q_1 \cap \cdots \cap Q_m$$

be an irredundant minimal primary decomposition of $\mathscr{I}_\infty(Q)$. By Theorem 2.3 we can assume that the $Q_1, \ldots, Q_m$ are unstable modules. Hence the $\mathscr{P}^*$-invariant primary ideal $(\mathscr{I}_\infty(Q) : M)$ can be written as an intersection of $\mathscr{P}^*$-invariant primary ideals

$$(\mathscr{I}_\infty(Q) : M) = \bigcap_{i=1}^{m} (Q_i : M).$$

By irredundancy and minimality we obtain $m = 1$ as desired. □

References

[1] S. Balcerzyk, T. Józefiak, Commutative Noetherian and Krull Rings, Ellis Horwood, Chichester, 1989.
[2] L.E. Dickson, A fundamental system of invariants of the general modular linear group with a solution of the form problem, Trans. AMS 12 (1911) 75–98.
[3] S.-P. Lam, Unstable algebras over the Steenrod algebra and cohomology of classifying spaces, Proc. Conf. on Algebraic Topology, Aarhus, 1982, Springer, Heidelberg, 1983.

[4] W.S. Massey, F.P. Peterson, The cohomology structure of certain fibre spaces I, Topology 4 (1965) 47–65.
[5] M.D. Neusel, Integral extensions of unstable algebras over the Steenrod algebra, Forum Math. 12 (2000) 155–166.
[6] M.D. Neusel, Inverse invariant theory and Steenrod operations, Memoirs of AMS, No. 692, Vol. 146, AMS, Providence, RI, 2000.
[7] M.D. Neusel, The Lasker–Noether theorem in the category $\mathscr{U}(\mathrm{H}^*)$, Applications, in preparation.
[8] M.D. Neusel, L. Smith, The Lasker–Noether theorem for $\mathscr{P}^*$-invariant ideals, Forum Math. 10 (1998) 1–18.
[9] E. Noether, Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Characteristik p, Nachr. v. d. Ges. d. Wiss. zu Göttingen (1926) 28–35.
[10] L. Smith, Polynomial Invariants of Finite Groups, 2nd corrected printing, Peters, Wellesley, MA, 1997.
[11] L. Smith, Polynomial Invariants of Finite Groups. A survey of recent developments, Bull. AMS 34 (1997) 211–250.
[12] O. Zariski, P. Samuel, Commutative Algebra, Vol. I, van Nostrand, Princeton, NJ, 1958.